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The constrained multiple-sets split feasibility problem and its projection algorithms

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Abstract
The projection algorithms for solving the constrained multiple-sets split feasibility problem are presented. The strong convergence results of the algorithms are given under some mild conditions. Especially, the minimum norm solution of the constrained multiple-sets split feasibility problem can be found.

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let $C_1, C_2, \dots, C_N$ be N nonempty closed convex subsets of H_1 and let $Q_1, Q_2, \dots, Q_M$ be M nonempty closed convex subsets of H_2. Let $A : H_1 \to H_2$ be a bounded linear operator. The multiple-sets split feasibility problem is formulated as follows:

$$\text{Find an } x \in \bigcap_{i=1}^{N} C_i \text{ such that } Ax \in \bigcap_{j=1}^{M} Q_j. \tag{1.1}$$

A special case If $N = M = 1$, then the multiple-sets split feasibility problem is reduced to the split feasibility problem which is formulated as finding a point x with the property

$$x \in C \quad \text{and} \quad Ax \in Q.$$

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the multiple-sets split feasibility problem and the split feasibility problem can be used to model the intensity-modulated radiation therapy [3–6]. Various algorithms have been invented to solve the multiple-sets split feasibility problem and the split feasibility problem, see, *e.g.*, [7–24] and references therein.

The popular algorithm that solves the multiple-sets split feasibility problem and the split feasibility problem is Byrne's CQ algorithm [11] which is found to be a gradient-projection method in convex minimization. Motivated by this idea, in this paper, we present the composite projection algorithms for solving the constrained multiple-sets split feasibility problem. The strong convergence results of the algorithms are given under some mild conditions. Especially, the minimum norm solution of the constrained multiple-sets split feasibility problem can be found.

2 Preliminaries

2.1 Concepts

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively, and let Ω be a nonempty closed convex subset of H. Recall that the (nearest point or metric) projection from H onto Ω, denoted by P_Ω, is defined in such a way that, for each $x \in H$, $P_\Omega(x)$ is the unique point in Ω with the property

$$\left\|x - P_\Omega(x)\right\| = \min\left\{\|x-y\| : y \in \Omega\right\}.$$

It is known that P_Ω satisfies

$$\left\langle x-y, P_\Omega(x) - P_\Omega(y)\right\rangle \geq \left\|P_\Omega(x) - P_\Omega(y)\right\|^2, \quad \forall x, y \in H.$$

Moreover, P_Ω is characterized by the following properties:

$$\left\langle x - P_\Omega(x), y - P_\Omega(x)\right\rangle \leq 0$$

for all $x \in H$ and $y \in \Omega$.

We also recall that a mapping $f : \Omega \to H$ is said to be ρ-contractive if $\|Tx - Ty\| \leq \rho\|x - y\|$ for some constant $\rho \in [0,1)$ and for all $x, y \in \Omega$. A mapping $T : \Omega \to \Omega$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \Omega$. A mapping T is called averaged if $T = (1-\delta)I + \delta U$, where $\delta \in (0,1)$ and $U : \Omega \to \Omega$ is nonexpansive. In this case, we also say that T is δ-averaged. A bounded linear operator B is said to be strongly positive on H if there exists a constant $\alpha > 0$ such that

$$\langle Bx, x\rangle \geq \alpha\|x\|^2, \quad \forall x \in H.$$

Let A be an operator with domain $D(A)$ and range $R(A)$ in H.

(i) A is monotone if for all $x, y \in D(A)$,

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

(ii) Given a number $\nu > 0$, A is said to be ν-inverse strongly monotone (ν-ism) (or co-coercive) if

$$\langle Ax - Ay, x - y\rangle \geq \nu\|Ax - Ay\|^2, \quad x, y \in H.$$

It is easily seen that a projection P_Ω is a 1-ism and hence P_Ω is $\frac{1}{2}$-averaged.

We will need to use the following notation:

- $\mathrm{Fix}(T)$ stands for the set of fixed points of T;
- $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;
- $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

2.2 Mathematical model

Now, we consider the mathematical model of the multiple-sets split feasibility problem. Let $x \in C_1$. Assume that $Ax \in Q_1$. Then we get $(I - P_{Q_1})Ax = 0$, which implies $\gamma A^*(I -$

$P_{Q_1})Ax = 0$, hence x satisfies the fixed point equation $x = (I - \gamma A^*(I - P_{Q_1})A)x$. At the same time, note that $x \in C_1$. Thus,

$$x = P_{C_1}\bigl(I - \gamma A^*(I - P_{Q_1})A\bigr)x.$$

Now, we know x solves the split feasibility problem if and only if x solves the above fixed point equation. This result reminds us that the multiple-sets split feasibility problem is equivalent to a common fixed point problem of finitely many nonexpansive mappings. On the other hand, x solves the multiple-sets split feasibility problem implies that x satisfies two properties:

(i) the distance from x to each C_i is zero and
(ii) the distance from Ax to each Q_j is also zero.

First, we consider the following proximity function:

$$g(x) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{M} \beta_j \|Ax - P_{Q_j}Ax\|^2,$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are positive real numbers, and P_{C_i} and P_{Q_j} are the metric projections onto C_i and Q_j, respectively. It is clear that the proximity function g is convex and differentiable with the gradient

$$\nabla g(x) = \sum_{i=1}^{N} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax.$$

We can check that the gradient $\nabla g(x)$ is L-Lipschitz continuous with constant

$$L = \sum_{i=1}^{N} \alpha_i + \sum_{j=1}^{M} \beta_j \|A\|^2.$$

Note that x^* is a solution of the multiple-sets split feasibility problem (1.1) if and only if $g(x^*) = 0$. Since $g(x) \geq 0$ for all $x \in H_1$, a solution of the multiple-sets split feasibility problem (1.1) is a minimizer of g over any closed convex subset, with minimum value of zero. This motivates us to consider the following minimization problem:

$$\min_{x \in \Omega} g(x), \tag{2.1}$$

where Ω is a closed convex subset of H_1 whose intersection with the solution set of the multiple-sets split feasibility problem is nonempty, and get a solution of the so-called constrained multiple-sets split feasibility problem

$$x^* \in \Omega \text{ such that } x^* \text{ solves (1.1)}. \tag{2.2}$$

2.3 The well-known lemmas

The following lemmas will be helpful for our main results in the next section.

Lemma 2.1 [25] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} =$*

$(1-\beta_n)z_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leq 0$. *Then* $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.2 [26] *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T: K \to K$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Then T is demiclosed on K, i.e., if $x_n \rightharpoonup x \in K$ weakly and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3 [27] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*
(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(2) $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $C_1, C_2, \ldots, C_N$ be N nonempty closed convex subsets of H_1 and let $Q_1, Q_2, \ldots, Q_M$ be M nonempty closed convex subsets of H_2. Let $A: H_1 \to H_2$ be a bounded linear operator. Assume that the multiple-sets split feasibility problem is consistent, *i.e.*, it is solvable. Now, we are devoted to solving the constrained multiple-set split feasibility problem (2.2).

For solving (2.2), we introduce the following iterative algorithm.

Algorithm 3.1 Let $f: H_1 \to H_1$ be a ρ-contraction. Let $B: H_1 \to H_1$ be a self-adjoint, strongly positive bounded linear operator with coefficient $\alpha > 0$. Let σ and γ be two constants such that $0 < \gamma < \frac{2}{L}$ and $0 < \sigma\rho < \alpha$. For arbitrary initial point $x_0 \in H_1$, we define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} x_{n+1} = {} & P_\Omega\left(I - \gamma\left(\sum_{i=1}^{N}\alpha_i(I-P_{C_i}) + \sum_{j=1}^{M}\beta_j A^*(I-P_{Q_j})A\right)\right) \\ & \times P_\Omega\big(\xi_n\sigma f + (I-\xi_n B)\big)x_n, \end{aligned} \tag{3.1}$$

for all $n \geq 0$, where $\{\xi_n\}$ is a real sequence in $(0,1)$.

Fact 3.2 The mapping $I - \gamma(\sum_{i=1}^{N}\alpha_i(I-P_{C_i}) + \sum_{j=1}^{M}\beta_j A^*(I-P_{Q_j})A)$ is $\frac{\gamma L}{2}$-averaged.

In order to check Fact 3.2, we need the following lemmas.

Lemma 3.3 (Baillon-Haddad) [28] *If $h: H \to R$ has an L-Lipschitz continuous gradient ∇h, then ∇h is $\frac{1}{L}$-ism.*

Lemma 3.4 *Given $T: H \to H$ and let $V = I - T$ be the complement of T. Given also $S: H \to H$.*
(i) *T is nonexpansive if and only if V is $\frac{1}{2}$-inverse strongly monotone (in short, $\frac{1}{2}$-ism).*
(ii) *If S is ν-ism, then for $\gamma > 0$, γS is $\frac{\nu}{\gamma}$-ism.*
(iii) *S is averaged if and only if the complement $I - S$ is ν-ism for some $\nu > \frac{1}{2}$.*

Lemma 3.5 *Given operators $S, T, V: H \to H$.*
(i) *If $S = (1-\alpha)T + \alpha V$ for some $\alpha \in (0,1)$ and if T is averaged and V is nonexpansive, then S is averaged.*

(ii) *S is firmly nonexpansive if and only if the complement $I - S$ is firmly nonexpansive. If S is firmly nonexpansive, then S is averaged.*
(iii) *If $S = (1-\alpha)T + \alpha V$ for some $\alpha \in (0,1)$, T is firmly nonexpansive and V is nonexpansive, then S is averaged.*
(iv) *If S and T are both averaged, then the product* (*composite*) *ST is averaged.*

Proof of Fact 3.2 Since gradient $\nabla g(x) = \sum_{i=1}^{N} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax$ has an L-Lipschitz constant $L = \sum_{i=1}^{N} \alpha_i + \sum_{j=1}^{M} \beta_j \|A\|^2$, by Lemma 3.4, ∇g is $\frac{1}{L}$-ism and $\gamma(\sum_{i=1}^{N} \alpha_i (I - P_{C_i}) + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A)$ is $\frac{1}{\gamma L}$-ism. Again, from Lemma 3.4(iii), we deduce that $I - \gamma(\sum_{i=1}^{N} \alpha_i (I - P_{C_i}) + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A)$ is $\frac{\gamma L}{2}$-averaged. □

Now, we prove the convergence of the sequence $\{x_n\}$.

Theorem 3.6 *Suppose that $S \neq \emptyset$. Assume that the sequence $\{\xi_n\}$ satisfies the control conditions*:
(i) $\lim_{n\to\infty} \xi_n = 0$ *and*
(ii) $\sum_{n=0}^{\infty} \xi_n = \infty$.
Then the sequence $\{x_n\}$ generated by (3.1) *converges to a solution x^* of* (2.2), *where x^* also solves the following VI*:

$$x^* \in S \text{ such that } \langle \sigma f(x^*) - Bx^*, \tilde{x} - x^* \rangle \le 0 \quad \text{for all } \tilde{x} \in S, \tag{3.2}$$

where S is the set of solutions of (2.2).

Proof Let $x^* \in S$. Since B is strongly positive bounded linear operator with coefficient $\alpha > 0$, we have $\|I - \xi_n B\| \le 1 - \alpha\xi_n$ (without loss of generality, we may assume $\xi_n \le \frac{1}{\alpha}$). Thus, by (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad= \Bigg\| P_\Omega \left(I - \gamma \left(\sum_{i=1}^{N} \alpha_i (I - P_{C_i}) + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A \right) \right) \\
&\qquad \times P_\Omega \big(\xi_n \sigma f + (I - \xi_n B)\big) x_n - x^* \Bigg\| \\
&\quad\le \big\| \xi_n \sigma f(x_n) + (I - \xi_n B)x_n - x^* \big\| \\
&\quad\le \xi_n \sigma \big\| f(x_n) - f(x^*) \big\| + \|I - \xi_n B\| \big\| x_n - x^* \big\| + \xi_n \big\| \sigma f(x^*) - Bx^* \big\| \\
&\quad\le \xi_n \sigma\rho \big\| x_n - x^* \big\| + (1 - \xi_n \alpha) \big\| x_n - x^* \big\| + \xi_n \big\| \sigma f(x^*) - Bx^* \big\| \\
&\quad= \big[1 - (\alpha - \sigma\rho)\xi_n\big] \big\| x_n - x^* \big\| + (\alpha - \sigma\rho)\xi_n \big\| f(x^*) - Bx^* \big\| / (\alpha - \sigma\rho).
\end{aligned}$$

An induction yields

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le \max\left\{ \|x_n - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\alpha - \sigma\rho} \right\} \\
&\le \max\left\{ \|x_0 - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\alpha - \sigma\rho} \right\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded.

It is well-known that the metric projection P_Ω is firmly nonexpansive, hence averaged. By Fact 3.2, $I-\gamma(\sum_{i=1}^N \alpha_i(I-P_{C_i})+\sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A)$ is $\frac{\gamma L}{2}$-averaged. From Lemma 3.5, the composite of three averaged mappings is averaged. So, $P_\Omega(I-\gamma(\sum_{i=1}^N \alpha_i(I-P_{C_i})+\sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A))P_\Omega$ is an averaged mapping. Thus, there must exist a positive constant $\delta\in(0,1)$ such that

$$P_\Omega\left(I-\gamma\left(\sum_{i=1}^N \alpha_i(I-P_{C_i})+\sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A\right)\right)P_\Omega=(1-\delta)I+\delta U,$$

where U is a nonexpansive mapping. Set $y_n=\xi_n\sigma f(x_n)+(I-\xi_n B)x_n$ for all $n\geq 0$. Then we have

$$\begin{aligned} x_{n+1} &= \big((1-\delta)I+\delta U\big)\big(\xi_n\sigma f(x_n)+(I-\xi_n B)x_n\big) \\ &= (1-\delta)x_n+\xi_n(1-\delta)\big(\sigma f(x_n)-Bx_n\big)+\delta U y_n \\ &= (1-\delta)x_n+\delta\left(\frac{1-\delta}{\delta}\xi_n\big(\sigma f(x_n)-Bx_n\big)+Uy_n\right) \\ &= (1-\delta)x_n+\delta z_n, \end{aligned}$$

where

$$z_n=\frac{(1-\delta)\xi_n}{\delta}\big(\sigma f(x_n)-Bx_n\big)+Uy_n.$$

By virtue of $\xi_n\to 0$ (as $n\to\infty$) and the boundedness of the sequences $\{f(x_n)\}$ and $\{Bx_n\}$, we firstly observe that

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\xi_n\big\|\sigma f(x_n)-Bx_n\big\|=0,$$

and

$$\lim_{n\to\infty}\|z_n-Uy_n\|=\lim_{n\to\infty}\frac{(1-\delta)\xi_n}{\delta}\big\|\sigma f(x_n)-Bx_n\big\|=0.$$

Next, we estimate $\|z_{n+1}-z_n\|$. Note that

$$z_{n+1}-z_n=\frac{(1-\delta)\xi_{n+1}}{\delta}\big(\sigma f(x_{n+1})-Bx_{n+1}\big)+Uy_{n+1}-\frac{(1-\delta)\xi_n}{\delta}\big(\sigma f(x_n)-Bx_n\big)-Uy_n.$$

It follows that

$$\begin{aligned} \|z_{n+1}-z_n\| &\leq \frac{1-\delta}{\delta}\big(\xi_{n+1}\big\|\sigma f(x_{n+1})-Bx_{n+1}\big\|+\xi_n\big\|\sigma f(x_n)-Bx_n\big\|\big)+\|Uy_{n+1}-Uy_n\| \\ &\leq \frac{1-\delta}{\delta}\big(\xi_{n+1}\big\|\sigma f(x_{n+1})-Bx_{n+1}\big\|+\xi_n\big\|\sigma f(x_n)-Bx_n\big\|\big)+\|y_{n+1}-y_n\|. \end{aligned}$$

Since $y_{n+1}-y_n=\xi_{n+1}\sigma f(x_{n+1})+(I-\xi_{n+1}B)x_{n+1}-\xi_n\sigma f(x_n)-(I-\xi_n B)x_n$, we get

$$\begin{aligned} \|z_{n+1}-z_n\| \leq{}& \big\|\xi_{n+1}\sigma f(x_{n+1})+(I-\xi_{n+1}B)x_{n+1}-\xi_n\sigma f(x_n)-(I-\xi_n B)x_n\big\| \\ &+\frac{1-\delta}{\delta}\big(\xi_{n+1}\big\|\sigma f(x_{n+1})-Bx_{n+1}\big\|+\xi_n\big\|\sigma f(x_n)-Bx_n\big\|\big) \end{aligned}$$

$$\begin{aligned}&\le \|x_{n+1}-x_n\| + \xi_{n+1}\big\|\sigma f(x_{n+1}) - Bx_{n+1}\big\| + \xi_n\big\|\sigma f(x_n) - Bx_n\big\| \\ &\quad + \frac{1-\delta}{\delta}\big(\xi_{n+1}\big\|\sigma f(x_{n+1}) - Bx_{n+1}\big\| + \xi_n\big\|\sigma f(x_n) - Bx_n\big\|\big).\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| &\le \xi_{n+1}\big\|\sigma f(x_{n+1}) - Bx_{n+1}\big\| + \xi_n\big\|\sigma f(x_n) - Bx_n\big\| \\ &\quad + \frac{1-\delta}{\delta}\big(\xi_{n+1}\big\|\sigma f(x_{n+1}) - Bx_{n+1}\big\| + \xi_n\big\|\sigma f(x_n) - Bx_n\big\|\big).\end{aligned}$$

Since $\lim_{n\to\infty}\xi_n = 0$ and the sequences $\{f(x_n)\}$, $\{Bx_n\}$ are bounded, we deduce

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\big) \le 0.$$

By Lemma 2.1, we get

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\|Ux_n - x_n\| \\ &\quad = \lim_{n\to\infty}\left\|P_\Omega\left(I - \gamma\left(\sum_{i=1}^{N}\alpha_i(I-P_{C_i}) + \sum_{j=1}^{M}\beta_j A^*(I-P_{Q_j})A\right)\right)P_\Omega(x_n) - x_n\right\| = 0.\end{aligned}$$

By the definition of the sequence $\{x_n\}$, we know that $x_n \in \Omega$. Hence, $P_\Omega(x_n) = x_n$. So,

$$\lim_{n\to\infty}\left\|P_\Omega\left(I - \gamma\left(\sum_{i=1}^{N}\alpha_i(I-P_{C_i}) + \sum_{j=1}^{M}\beta_j A^*(I-P_{Q_j})A\right)\right)x_n - x_n\right\| = 0.$$

Next we prove

$$\limsup_{n\to\infty}\big\langle\sigma f\big(x^*\big) - Bx^*, P_\Omega(y_n) - x^*\big\rangle \le 0.$$

In order to get this inequality, we need to prove the following:

$$\limsup_{n\to\infty}\big\langle\sigma f\big(x^*\big) - Bx^*, x_n - x^*\big\rangle \le 0,$$

where x^* is the unique solution of VI(3.2). For this purpose, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle\sigma f\big(x^*\big) - Bx^*, x_n - x^*\big\rangle = \lim_{i\to\infty}\big\langle\sigma f\big(x^*\big) - Bx^*, x_{n_i} - x^*\big\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $\tilde{x}$. Since

$P_\Omega(I-\gamma(\sum_{i=1}^N \alpha_i(I-P_{C_i})+\sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A))$ is nonexpansive, by Lemma 2.2, we have $x_{n_i} \rightharpoonup \tilde{x} \in \operatorname{Fix}(P_\Omega(I-\gamma(\sum_{i=1}^N \alpha_i(I-P_{C_i})+\sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A)))$. Therefore,

$$\begin{aligned}
\limsup_{n\to\infty}\langle\sigma f(x^*)-Bx^*, x_n-x^*\rangle &= \lim_{i\to\infty}\langle\sigma f(x^*)-Bx^*, x_{n_i}-x^*\rangle \\
&= \langle\sigma f(x^*)-Bx^*, \tilde{x}-x^*\rangle \le 0.
\end{aligned}$$

Since $\|x_n - P_\Omega(y_n)\| = \|P_\Omega(x_n)-P_\Omega(y_n)\| \le \|x_n-y_n\| \to 0$, we obtain

$$\limsup_{n\to\infty}\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle \le 0.$$

Note that

$$\|P_\Omega(y_n)-x^*\|^2 = \langle P_\Omega(y_n)-y_n, P_\Omega(y_n)-x^*\rangle + \langle y_n-x^*, P_\Omega(y_n)-x^*\rangle.$$

From the property of the metric P_Ω, we have $\langle P_\Omega(y_n)-y_n, P_\Omega(y_n)-x^*\rangle \le 0$. Hence,

$$\begin{aligned}
\|P_\Omega(y_n)-x^*\|^2 &\le \langle y_n-x^*, P_\Omega(y_n)-x^*\rangle \\
&= \langle \xi_n\sigma(f(x_n)-f(x^*)) + (I-\xi_n B)(x_n-x^*), P_\Omega(y_n)-x^*\rangle \\
&\quad + \xi_n\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle \\
&\le (\xi_n\sigma\|f(x_n)-f(x^*)\| + \|I-\xi_n B\|\|x_n-x^*\|)\|P_\Omega(y_n)-x^*\| \\
&\quad + \xi_n\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle \\
&\le (1-\xi_n(\alpha-\sigma\rho))\|x_n-x^*\|\|P_\Omega(y_n)-x^*\| \\
&\quad + \xi_n\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle \\
&\le \frac{1-\xi_n(\alpha-\sigma\rho)}{2}\|x_n-x^*\|^2 + \frac{1}{2}\|P_\Omega(y_n)-x^*\|^2 \\
&\quad + \xi_n\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle.
\end{aligned}$$

It follows that

$$\|P_\Omega(y_n)-x^*\|^2 \le [1-(\alpha-\sigma\rho)\xi_n]\|x_n-x^*\|^2 + 2\xi_n\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle.$$

Finally, we show that $x_n \to x^*$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\| P_\Omega\left(I-\gamma\left(\sum_{i=1}^N \alpha_i(I-P_{C_i}) + \sum_{j=1}^M \beta_j A^*(I-P_{Q_j})A\right)\right)P_\Omega(y_n)-x^*\right\|^2 \\
&\le \|P_\Omega(y_n)-x^*\|^2 \\
&\le [1-(\alpha-\sigma\rho)\xi_n]\|x_n-x^*\|^2 \\
&\quad + (\alpha-\sigma\rho)\xi_n\frac{2}{\alpha-\sigma\rho}\langle\sigma f(x^*)-Bx^*, P_\Omega(y_n)-x^*\rangle \\
&= (1-\gamma_n)\|x_n-x^*\|^2 + \delta_n,
\end{aligned}$$

where $\gamma_n = (\alpha - \sigma\rho)\xi_n$ and $\delta_n = (\alpha - \sigma\rho)\xi_n \frac{2}{\alpha-\sigma\rho}\langle \sigma f(x^*) - Bx^*, P_\Omega(y_n) - x^*\rangle$. Since $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} \frac{2}{\alpha-\sigma\rho}\langle \sigma f(x^*) - Bx^*, P_\Omega(y_n) - x^*\rangle \leq 0$, all conditions of Lemma 2.3 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$. This completes the proof. □

From (3.1) and Theorem 3.6, we can deduce easily the following results.

Algorithm 3.7 For an arbitrary initial point $x_0 \in H_1$, we define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} x_{n+1} = {} & P_\Omega\left(I - \gamma\left(\sum_{i=1}^{N} \alpha_i(I - P_{C_i}) + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right)\right) \\ & \times P_\Omega\bigl(\xi_n \sigma f(x_n) + (1 - \xi_n)x_n\bigr), \end{aligned} \tag{3.3}$$

for all $n \geq 0$, where $\{\xi_n\}$ is a real sequence in $(0,1)$.

Corollary 3.8 *Suppose that $S \neq \emptyset$. Assume that the sequence $\{\xi_n\}$ satisfies the conditions*
(i) $\lim_{n\to\infty} \xi_n = 0$ *and*
(ii) $\sum_{n=0}^{\infty} \xi_n = \infty$.
Then the sequence $\{x_n\}$ generated by (3.3) *converges to a point x^*, which solves the following variational inequality*:

$$x^* \in S \text{ such that } \bigl\langle \sigma f\bigl(x^*\bigr) - x^*, \tilde{x} - x^*\bigr\rangle \leq 0 \quad \text{for all } \tilde{x} \in S.$$

Algorithm 3.9 For an arbitrary initial point x_0, we define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_\Omega\left(I - \gamma\left(\sum_{i=1}^{N} \alpha_i(I - P_{C_i}) + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right)\right) P_\Omega\bigl((1 - \xi_n)x_n\bigr), \tag{3.4}$$

for all $n \geq 0$, where $\{\xi_n\}$ is a real sequence in $(0,1)$.

Corollary 3.10 *Suppose that $S \neq \emptyset$. Assume that the sequence $\{\xi_n\}$ satisfies the conditions*
(i) $\lim_{n\to\infty} \xi_n = 0$ *and*
(ii) $\sum_{n=0}^{\infty} \xi_n = \infty$.
Then the sequence $\{x_n\}$ generated by (3.4) *converges to a point $x^* \in S$ which is the minimum norm element in S.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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